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Hille and Nehari type criteria for third-order dynamic equations

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Abstract

In this paper, we extend the oscillation criteria that have been established by Hille [E. Hille, Non-oscillation theorems, Trans. Amer. Math. Soc. 64 (1948) 234–252] and Nehari [Z. Nehari, Oscillation criteria for second-order linear differential equations, Trans. Amer. Math. Soc. 85 (1957) 428–445] for second-order differential equations to third-order dynamic equations on an arbitrary time scale $\mathbb{T}$, which is unbounded above. Our results are essentially new even for third-order differential and difference equations, i.e., when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$. We consider several examples to illustrate our results.

Keywords: Oscillation; Third-order dynamic equations; Time scales

1. Introduction

The study of dynamic equations on time scales, which goes back to its founder Stefan Hilger [14], is an area of mathematics that has recently received a lot of attention. It has been created in order to unify the study of differential and difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations,

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while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies, and helps avoid proving results twice—once for differential equations and once again for difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which is a nonempty closed subset of the reals $\mathbb{R}$. In this way results not only related to the set of real numbers or set of integers but those pertaining to more general time scales are obtained.

The three most popular examples of calculus on time scales are differential calculus, difference calculus (see [17]), and quantum calculus (see Kac and Cheung [16]), i.e., when $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$ and $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t\colon\ t\in\mathbb{N}_0\}$, where $q>1$. Dynamic equations on a time scale have an enormous potential for applications such as in population dynamics. For example, it can model insect populations that are continuous while in season, die out in say winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population (see [4]). There are applications of dynamic equations on time scales to quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics. A recent cover story article in New Scientist [29] discusses several possible applications. The books on the subject of time scales by Bohner and Peterson [4,5] summarize and organize much of time scale calculus and some applications.

For completeness, we recall the following concepts related to the notion of time scales. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The *forward jump operator* and the *backward jump operator* are defined by:

$$\sigma(t):=\inf\{s\in\mathbb{T}\colon\ s>t\},\qquad \rho(t):=\sup\{s\in\mathbb{T}\colon\ s<t\},$$

where $\sup\emptyset=\inf\mathbb{T}$. A point $t\in\mathbb{T}$, is said to be *left-dense* if $\rho(t)=t$ and $t>\inf\mathbb{T}$, is *right-dense* if $\sigma(t)=t$, is *left-scattered* if $\rho(t)<t$ and *right-scattered* if $\sigma(t)>t$. A function $g:\mathbb{T}\to\mathbb{R}$ is said to be *right-dense continuous* (rd-continuous) provided g is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{\mathrm{rd}}(\mathbb{T})$. The *graininess function* μ for a time scale $\mathbb{T}$ is defined by $\mu(t):=\sigma(t)-t$, and for any function $f:\mathbb{T}\to\mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$.

Definition 1. Fix $t\in\mathbb{T}$ and let $x:\mathbb{T}\to\mathbb{R}$. Define $x^\Delta(t)$ to be the number (if it exists) with the property that given any $\epsilon>0$ there is a neighbourhood U of t with

$$\bigl|\bigl[x\bigl(\sigma(t)\bigr)-x(s)\bigr]-x^\Delta(t)\bigl[\sigma(t)-s\bigr]\bigr|\leqslant\epsilon\bigl|\sigma(t)-s\bigr|,\quad \text{for all } s\in U.$$

In this case, we say $x^\Delta(t)$ is the (*delta*) *derivative* of x at t and that x is (*delta*) *differentiable* at t.

We will frequently use the results in the following theorem which is due to Hilger [14].

Theorem 1. *Assume that $g:\mathbb{T}\to\mathbb{R}$ and let $t\in\mathbb{T}$.*

(i) *If g is differentiable at t, then g is continuous at t.*
(ii) *If g is continuous at t and t is right-scattered, then g is differentiable at t with*

$$g^\Delta(t)=\frac{g(\sigma(t))-g(t)}{\mu(t)}.$$

(iii) *If g is differentiable and t is right-dense, then*

$$g^{\Delta}(t) = \lim_{s\to t} \frac{g(t)-g(s)}{t-s}.$$

(iv) *If g is differentiable at t, then $g(\sigma(t)) = g(t) + \mu(t)g^{\Delta}(t)$.*

In this paper we will refer to the (delta) integral which we can define as follows:

Definition 2. If $G^{\Delta}(t) = g(t)$, then the *Cauchy* (*delta*) *integral* of g is defined by

$$\int_a^t g(s)\Delta s := G(t) - G(a).$$

It can be shown (see [4]) that if $g \in C_{\mathrm{rd}}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^{\Delta}(t) = g(t)$, $t \in \mathbb{T}$. For a more general definition of the delta integral see [4,5].

In the last few years, there has been increasing interest in obtaining sufficient conditions for the oscillation/nonoscillation of solutions of different classes of dynamic equations on time scales. We refer the reader to the papers [1–3,6,7,9–12,22–28] and references cited therein. In this paper, we are concerned with the oscillatory behavior of solutions of the third-order linear dynamic equation

$$x^{\Delta\Delta\Delta}(t) + p(t)x(t) = 0, \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$, where $p(t)$ is a positive real-valued rd-continuous function defined on $\mathbb{T}$. Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup\mathbb{T} = \infty$, and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty)\cap\mathbb{T}$. By a solution of (1.1) we mean a nontrivial real-valued functions $x(t) \in C_r^3[T_x,\infty)$, $T_x \geqslant t_0$ where C_r is the space of rd-continuous functions. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory in case there exists at least one oscillatory solution.

We note that, Eq. (1.1) in its general form covers several different types of differential and difference equations depending on the choice of the time scale $\mathbb{T}$. For example, if $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $\mu(t) = 0$, $x^{\Delta}(t) = x'(t)$, $\int_a^b f(t)\Delta t = \int_a^b f(t)\,dt$ and (1.1) becomes the third-order linear differential equation

$$x'''(t) + p(t)x(t) = 0. \tag{1.2}$$

If $\mathbb{T} = \mathbb{N}$, then $\sigma(t) = t+1$, $\mu(t) = 1$, $x^{\Delta}(t) = \Delta x(t) = x(t+1) - x(t)$, $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$ and (1.1) becomes the third-order difference equation

$$\Delta^3 x(t) + p(t)x(t) = 0. \tag{1.3}$$

If $\mathbb{T} = h\mathbb{Z}^+$, $h > 0$, then $\sigma(t) = t + h$, $\mu(t) = h$, $x^{\Delta}(t) = \Delta_h x(t) = \frac{x(t+h)-x(t)}{h}$, $\int_a^b f(t)\Delta t = \sum_{k=0}^{\frac{b-a-h}{h}} f(a+kh)h$ and (1.1) becomes the third-order difference equation

$$\Delta_h^3 x(t) + p(t)x(t) = 0. \tag{1.4}$$

If $\mathbb{T} = q^{\mathbb{N}} = \{t\colon\ t = q^k,\ k \in \mathbb{N},\ q > 1\}$, then $\sigma(t) = q\,t$, $\mu(t) = (q-1)t$, $x^{\Delta}(t) = \Delta_q x(t) = \frac{x(q\,t)-x(t)}{(q-1)\,t}$ (this is the so-called quantum derivative, see Kac and Cheung [16]), $\int_a^b f(t)\Delta t = \sum_{t\in(a,b)} f(t)\mu(t)$ and (1.1) becomes the third-order q-difference equation

$$\Delta_q^3 x(t) + p(t)x(t) = 0. \tag{1.5}$$

When $\mathbb{T} = \mathbb{N}_0^2 = \{t = n^2\colon\ n \in \mathbb{N}_0\}$, then $\sigma(t) = (\sqrt{t}+1)^2$ and $\mu(t) = 1 + 2\sqrt{t}$, $\Delta_N x(t) = \frac{x((\sqrt{t}+1)^2)-x(t)}{1+2\sqrt{t}}$, $\int_a^b f(t)\Delta t = \sum_{t\in(a,b)} f(t)\mu(t)$ and (1.4) becomes the third-order equation

$$\Delta_N^3 x(n) + p(n)x(n) = 0. \tag{1.6}$$

If $\mathbb{T} = \mathbb{T}_n = \{t_n\colon\ n \in \mathbb{N}_0\}$ where $\{t_n\}$ is the set of the harmonic numbers defined by

$$t_0 = 0, \qquad t_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{N}_0,$$

then $\sigma(t_n) = t_{n+1}$, $\mu(t_n) = \frac{1}{n+1}$, $x^{\Delta}(t_n) = \Delta_{t_n} x(t_n) = (n+1)x(t_n)$, $\int_a^b f(t)\Delta t = \sum_{t\in(a,b)} f(t)\mu(t)$ and (1.1) becomes the third-order difference equation

$$\Delta_{t_n}^3 x(t_n) + p(t_n)x(t_n) = 0. \tag{1.7}$$

Leighton [19] studied the oscillatory behavior of solutions of the second-order linear differential equation

$$x''(t) + p(t)x(t) = 0, \tag{1.8}$$

and showed that if

$$\int_{t_0}^{\infty} p(t)\,dt = \infty, \tag{1.9}$$

then every solution of Eq. (1.8) oscillates.

Hille [15] improved the condition (1.9) and proved that every solution of (1.8) oscillates if

$$\liminf_{t\to\infty} t \int_t^{\infty} p(s)\,ds > \frac{1}{4}. \tag{1.10}$$

Nehari [21] by a different approach proved that if

$$\liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} s^2 p(s)\,ds > \frac{1}{4}, \tag{1.11}$$

then every solution of (1.8) oscillates.

The oscillatory behavior of the corresponding third-order equation (1.2) has been studied by a number of authors including Hanan [13], Lazer [18] and Mehri [20]; and various well-known integral and Kneser-type tests exist. Mehri [20] extended the result of Leighton [19] and proved that (1.2) is oscillatory if and only if (1.9) holds. But one can easily see that the condition (1.9) cannot be applied to the cases when $p(t) = \frac{\beta}{t^2}$ and $p(t) = \frac{\beta}{t^3}$ for some $\beta > 0$. Hanan [13] improved the condition (1.9) for Eq. (1.2) and showed that if

$$\int_{t_0}^{\infty} t^2 p(t)\,dt < \infty, \tag{1.12}$$

then (1.2) is nonoscillatory. A corollary of a result of Lazer [18, Theorem 3.1] implies that (1.2) is oscillatory in case

$$\int_{t_0}^{\infty} t^{1+\delta} p(t)\,dt = \infty, \quad \text{for some } 0 < \delta < 1, \tag{1.13}$$

which improves the condition (1.9). By comparison with the Euler–Cauchy equation it has been shown that (cf. Erbe [8]), if

$$\limsup_{t\to\infty} t^3 p(t) < \frac{2}{3\sqrt{3}}, \tag{1.14}$$

then (1.2) is nonoscillatory and if

$$\liminf_{t\to\infty} t^3 p(t) > \frac{2}{3\sqrt{3}}, \tag{1.15}$$

then (1.2) is oscillatory.

The natural question now is: *Do the oscillation conditions* (1.10) *and* (1.11) *due to Hille and Nehari for second-order differential equations extend to third-order linear dynamic equations on time scales*?

The purpose of this paper is to give an affirmative answer to this question. We will establish new oscillation criteria for (1.1) which guarantee that every solution oscillates or converges to zero. Our results improve the oscillation condition (1.9) and (1.13) that has been established by Mehri [20] and Lazer [18]. The results are essentially new for Eqs. (1.3)–(1.7). Some examples which dwell upon the importance of our main results are given. To the best of the authors' knowledge this approach for the investigation of the oscillatory behavior of solutions of (1.1) has not been studied before.

2. Main results

Before stating our main results, we begin with the following lemma which is extracted from [10] (also see [11]).

Lemma 1. *Suppose that $x(t)$ is an eventually positive solution of* (1.1). *Then there are only the following two cases for $t \geqslant t_1$ sufficiently large*:

(I) $x(t) > 0$, $x^{\Delta}(t) > 0$, $x^{\Delta\Delta}(t) > 0$,

or

(II) $x(t) > 0$, $x^{\Delta}(t) < 0$, $x^{\Delta\Delta}(t) > 0$.

Lemma 2. *Assume that*

$$\int_{t_0}^{\infty} p(s)\Delta s = \infty,$$

and let $x(t)$ be a solution of (1.1), *then $x(t)$ is oscillatory or*

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x^{\Delta}(t) = \lim_{t\to\infty} x^{\Delta\Delta}(t) = 0.$$

Proof. Assume the contrary and let $x(t)$ be a nonoscillatory solution which may be assumed to be positive for $t \in [t_0, \infty)_{\mathbb{T}}$. Then $x^{\Delta\Delta\Delta}(t) = -p(t)x(t) < 0$; hence $x^{\Delta\Delta}(t)$ is decreasing. If $x^{\Delta}(t) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$, then $x(t)$ is increasing and

$$x^{\Delta\Delta}(t) = x^{\Delta\Delta}(t_0) - \int_{t_0}^{t} p(s)x(s)\Delta s \leqslant x^{\Delta\Delta}(t_0) - x(t_0)\int_{t_0}^{t} p(s)\Delta s.$$

This implies that $\lim_{t\to\infty} x^{\Delta\Delta}(t) = -\infty$ which is a contradiction by Lemma 1. Now assume there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$, $t_1 \geqslant 1$, such that $x^{\Delta}(t) < 0$ and by Lemma 1 we may also assume $x^{\Delta\Delta}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Since $x^{\Delta}(t) < 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$, then $x(t)$ is decreasing and there are two cases:

Case 1. $\lim_{t\to\infty} x(t) = \alpha > 0$. Then $x(t) \geqslant \alpha$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Multiplying (1.1) by $\sigma(t)$ and integrating from t_1 to t we have

$$tx^{\Delta\Delta}(t) - t_1x^{\Delta\Delta}(t_1) - x^{\Delta}(t) + x^{\Delta}(t_1) + \int_{t_1}^{t} \sigma(s)p(s)x(s)\Delta s = 0.$$

It follows that

$$A = t_1x^{\Delta\Delta}(t_1) - x^{\Delta}(t_1) = tx^{\Delta\Delta}(t) - x^{\Delta}(t) + \int_{t_1}^{t} \sigma(s)p(s)x(s)\Delta s$$

$$\geqslant \alpha\int_{t_1}^{t} \sigma(s)p(s)\Delta s \geqslant \alpha\int_{t_1}^{t} p(s)\Delta s,$$

which is a contradiction.

Case 2. $\lim_{t\to\infty} x(t) = 0$. From the fact that $x^{\Delta\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$ it follows that $x^{\Delta}(t)$ is increasing and $\lim_{t\to\infty} x^{\Delta}(t) = \beta$ where $-\infty < \beta \leqslant 0$. This implies that $x^{\Delta}(t) \leqslant \beta$ for all $t \in [t_1, \infty)_{\mathbb{T}}$, and hence $x(t_1) \geqslant x(t) - \beta(t - t_1)$ which is impossible for $\beta < 0$. Therefore $\lim_{t\to\infty} x^{\Delta}(t) = 0$. Now $x^{\Delta\Delta\Delta}(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$ implies that $x^{\Delta\Delta}(t)$ is decreasing and $\lim_{t\to\infty} x^{\Delta\Delta}(t) = \gamma$ where $0 \leqslant \gamma < \infty$. This implies that $x^{\Delta}(t_1) \leqslant x^{\Delta}(t) - \gamma(t - t_1)$ which again is impossible for $\gamma > 0$, and hence $\gamma = 0$. This completes the proof. □

Remark 1. If we assume that $\int_{t_0}^{\infty} p(t)\Delta t < \infty$, then it can easily be shown that the existence of a solution of (1.1) satisfying case (II) of Lemma 1 is incompatible with $\int_{t_0}^{\infty}\int_{t}^{\infty} p(s)\Delta s\Delta t = \infty$. In Lemma 3 we consider what happens if there is a solution of (1.1) satisfying case (II) in Lemma 1 if $\int_{t_0}^{\infty} p(t)\Delta t < \infty$, $\int_{t_0}^{\infty}\int_{t}^{\infty} p(s)\Delta s\Delta t < \infty$, and (2.1) below holds.

Lemma 3. *Assume that $x(t)$ is a solution of* (1.1) *which satisfies case* (II) *of Lemma* 1. *If*

$$\int_{t_0}^{\infty}\int_{z}^{\infty}\int_{u}^{\infty} p(s)\Delta s\Delta u\Delta z = \infty, \tag{2.1}$$

then $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be a solution of (1.1) such that case (II) of Lemma 1 holds for $t \in [t_1, \infty)_{\mathbb{T}}$. Since $x(t)$ is positive and decreasing, $\lim_{t\to\infty} x(t) := l \geqslant 0$. Assume that $\lim_{t\to\infty} x(t) = l > 0$.

Integrating both sides of Eq. (1.1) from t to ∞, we get $x^{\Delta\Delta}(t) \geqslant \int_t^\infty p(s)x(s)\Delta s$. Integrating again from t to ∞, we have $-x^{\Delta}(t) \geqslant \int_t^\infty \int_u^\infty p(s)x(s)\Delta s\,\Delta u$. Integrating again from t_0 to ∞, we obtain

$$x(t_0) \geqslant \int_{t_0}^{\infty}\int_{z}^{\infty}\int_{u}^{\infty} p(s)x(s)\Delta s\,\Delta u\,\Delta z.$$

Since $x(t) \geqslant l$, we see that

$$x(t_0) \geqslant l\int_{t_0}^{\infty}\int_{z}^{\infty}\int_{u}^{\infty} p(s)\Delta s\,\Delta u\,\Delta z.$$

This contradicts (2.1). Thus $l = 0$ and the proof is complete. □

In [4, Section 1.6] the Taylor monomials $\{h_n(t,s)\}_{n=0}^{\infty}$ are defined recursively by

$$h_0(t,s) = 1, \quad h_{n+1}(t,s) = \int_s^t h_n(\tau,s)\Delta\tau, \quad t,s \in \mathbb{T},\ n \geqslant 1.$$

It follows [4, Section 1.6] that $h_1(t,s) = t - s$ for any time scale, but simple formulas in general do not hold for $n \geqslant 2$. However, if $\mathbb{T} = \mathbb{R}$, then $h_n(t,s) = \frac{(t-s)^n}{n!}$; if $\mathbb{T} = \mathbb{N}_0$, then $h_n(t,s) = \frac{(t-s)^{\underline{n}}}{n!}$, where $t^{\underline{n}} = t(t-1)\cdots(t-n+1)$ is the so-called falling (factorial) function (cf. Kelley and Peterson [17]); and if $\mathbb{T} = q^{\mathbb{N}_0}$, then $h_n(t,s) = \prod_{\nu=0}^{n-1} \frac{t-q^{\nu}s}{\sum_{\mu=0}^{\nu} q^{\mu}}$. We will use these Taylor monomials in the rest of this paper.

Lemma 4. *Assume x satisfies*

$$x(t) > 0, \quad x^{\Delta}(t) > 0, \quad x^{\Delta\Delta}(t) > 0, \quad x^{\Delta\Delta\Delta} \leqslant 0, \quad t \in [T,\infty)_{\mathbb{T}}.$$

Then

$$\liminf_{t\to\infty} \frac{t x(t)}{h_2(t,t_0)x^{\Delta}(t)} \geqslant 1. \tag{2.2}$$

Proof. Let

$$G(t) := (t-T)x(t) - h_2(t,T)x^{\Delta}(t).$$

Then $G(T) = 0$ and

$$\begin{aligned}
G^{\Delta}(t) &= \big(\sigma(t) - T\big)x^{\Delta}(t) + x(t) - h_2\big(\sigma(t),T\big)x^{\Delta\Delta}(t) - (t-T)x^{\Delta}(t)\\
&= \mu(t)x^{\Delta}(t) + x(t) - h_2\big(\sigma(t),T\big)x^{\Delta\Delta}(t)\\
&= x^{\sigma}(t) - h_2\big(\sigma(t),T\big)x^{\Delta\Delta}(t)\\
&= x^{\sigma}(t) - \left(\int_T^{\sigma(t)} (\tau - T)\Delta\tau\right)x^{\Delta\Delta}(t).
\end{aligned}$$

By Taylor's Theorem [4, Theorem 1.113]

$$
\begin{aligned}
x^\sigma(t) &= x(T) + h_1\big(\sigma(t), T\big)x^\Delta(T) + \int_T^{\sigma(t)} h_1\big(\sigma(t), \sigma(\tau)\big)x^{\Delta\Delta}(\tau)\Delta\tau \\
&\geqslant x(T) + h_1\big(\sigma(t), T\big)x^\Delta(T) + x^{\Delta\Delta}(t)\int_T^{\sigma(t)} h_1\big(\sigma(t), \sigma(\tau)\big)\Delta\tau,
\end{aligned}
$$

since $x^{\Delta\Delta}(t)$ is nonincreasing. It would follow that $G^\Delta(t) > 0$ on $[T,\infty)_{\mathbb{T}}$ provided we can prove that

$$
\int_T^{\sigma(t)} h_1\big(\sigma(t), \sigma(\tau)\big)\Delta\tau = \int_T^{\sigma(t)} (t-\tau)\Delta\tau.
$$

To see this, we get by using the integration by parts formula [4, Theorem 1.77]

$$
\int_a^b f^\sigma(\tau)g^\Delta(\tau)\Delta\tau = \big[f(\tau)g(\tau)\big]_a^b - \int_a^b f^\Delta(\tau)g(\tau)\Delta\tau,
$$

and hence

$$
\begin{aligned}
\int_T^{\sigma(t)} h_1\big(\sigma(t), \sigma(\tau)\big)\Delta\tau &= \int_T^{\sigma(t)} \big(\sigma(t) - \sigma(\tau)\big)\Delta\tau \\
&= \big[\big(\sigma(t)-\tau\big)(\tau - T)\big]_{\tau=T}^{\tau=\sigma(t)} - \int_T^{\sigma(t)} (-1)(\tau - T)\Delta\tau \\
&= \int_T^{\sigma(t)} (\tau - T)\Delta\tau,
\end{aligned}
$$

which is the desired result. Hence $G^\Delta(t) > 0$ on $[T,\infty)_{\mathbb{T}}$. Since $G(T) = 0$ we get that $G(t) > 0$ on $(T,\infty)_{\mathbb{T}}$. This implies that

$$
\frac{(t-T)x(t)}{h_2(t,T)x^\Delta(t)} > 1, \quad t \in (T,\infty)_{\mathbb{T}}.
$$

Therefore, since

$$
\frac{tx(t)}{h_2(t,t_0)x^\Delta(t)} = \frac{(t-T)x(t)}{h_2(t,T)x^\Delta(t)} \cdot \frac{t}{t-T} \cdot \frac{h_2(t,T)}{h_2(t,t_0)},
$$

and since

$$
\lim_{t\to\infty} \frac{t}{t-T} = 1 = \lim_{t\to\infty} \frac{h_2(t,T)}{h_2(t,t_0)},
$$

we get that

$$
\liminf_{t\to\infty} \frac{tx(t)}{h_2(t,t_0)x^\Delta(t)} \geqslant 1. \qquad \square
$$

In the next result we will use the function $\Psi(t)$ defined by

$$\Psi(t) := \frac{h_2(t, t_0)}{\sigma(t)}.$$

Lemma 5. *Let x be a solution of* (1.1) *satisfying part* (I) *of Lemma* 1 *for $t \in [t_0, \infty)_{\mathbb{T}}$ and make the Riccati substitution*

$$w(t) = \frac{x^{\Delta\Delta}(t)}{x^{\Delta}(t)}.$$

Then

$$w^{\Delta}(t) + \frac{x(t)}{x^{\Delta\sigma}(t)} p(t) + \frac{w^2(t)}{1 + \mu(t)w(t)} = 0, \tag{2.3}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$. Furthermore given any $0 < k < 1$, there is a $T_k \in [t_0, \infty)_{\mathbb{T}}$ such that

$$w^{\Delta}(t) + k\Psi(t)p(t) + k\frac{t}{\sigma(t)} w^2(t) \leqslant 0, \tag{2.4}$$

$$w^{\Delta}(t) + k\Psi(t)p(t) + w(t)w^{\sigma}(t) \leqslant 0, \tag{2.5}$$

and

$$w^{\Delta}(t) + k\Psi(t)p(t) + \frac{w^2(t)}{1 + \mu(t)w(t)} \leqslant 0, \tag{2.6}$$

hold for $t \in [T_k, \infty)_{\mathbb{T}}$.

Proof. Let x be as in the statement of this lemma. Then by the quotient rule [4, Theorem 1.20] we have

$$\begin{aligned} w^{\Delta}(t) &= \left(\frac{x^{\Delta\Delta}}{x^{\Delta}}\right)^{\Delta} \\ &= \frac{x^{\Delta}(t)x^{\Delta\Delta\Delta}(t) - (x^{\Delta\Delta})^2}{x^{\Delta}(t)x^{\Delta\sigma}(t)} \\ &= -\frac{x^{\Delta}(t)p(t)x(t) - (x^{\Delta\Delta})^2}{x^{\Delta}(t)x^{\Delta\sigma}(t)} \\ &= -\frac{x(t)}{x^{\Delta\sigma}(t)} p(t) - \frac{x^{\Delta\Delta}(t)}{x^{\Delta\sigma}(t)} w(t). \end{aligned} \tag{2.7}$$

But

$$\begin{aligned} \frac{x^{\Delta\Delta}(t)}{x^{\Delta\sigma}(t)} w(t) &= \frac{x^{\Delta\Delta}(t)}{x^{\Delta}(t)} \cdot \frac{x^{\Delta}(t)}{x^{\Delta\sigma}(t)} w(t) \\ &= w^2(t) \cdot \frac{x^{\Delta}(t)}{x^{\Delta}(t) + \mu(t)x^{\Delta\Delta}(t)} \\ &= \frac{w^2(t)}{1 + \mu(t)w(t)}, \end{aligned} \tag{2.8}$$

so we get that (2.3) holds.

Next consider the coefficient of $p(t)$ in (2.3). Notice that

$$\frac{x(t)}{x^{\Delta\sigma}(t)} = \frac{x(t)}{x^{\Delta}(t)} \cdot \frac{x^{\Delta}(t)}{x^{\Delta\sigma}(t)}.$$

Now since

$$\liminf_{t\to\infty} \frac{tx(t)}{h_2(t,t_0)x^{\Delta}(t)} \geqslant 1,$$

given $0 < k < 1$, there is an $S_k \in [t_0,\infty)_{\mathbb{T}}$ such that

$$\frac{x(t)}{x^{\Delta}(t)} \geqslant \sqrt{k}\,\frac{h_2(t,t_0)}{t}, \quad t \in [S_k,\infty)_{\mathbb{T}}.$$

Also, since $x^{\Delta\sigma}(t) = x^{\Delta}(t) + \mu(t)x^{\Delta\Delta}(t)$ we have

$$\frac{x^{\Delta\sigma}(t)}{x^{\Delta}(t)} = 1 + \mu(t)\frac{x^{\Delta\Delta}(t)}{x^{\Delta}(t)},$$

and since $x^{\Delta\Delta\Delta}(t) = -p(t)x(t) < 0$, $x^{\Delta\Delta}(t)$ is decreasing and so

$$x^{\Delta}(t) = x^{\Delta}(t_1) + \int_{t_1}^{t} x^{\Delta\Delta}(\tau)\Delta\tau \geqslant x^{\Delta}(t_1) + x^{\Delta\Delta}(t)(t-t_1) > x^{\Delta\Delta}(t)(t-t_1),$$

for all $t > t_1 \geqslant S_k$. It follows that there is a $T_k \in [S_k,\infty)_{\mathbb{T}}$ such that

$$\frac{x^{\Delta}(t)}{x^{\Delta\Delta}(t)} \geqslant (t-t_1) \geqslant \sqrt{k}\,t$$

for $t \in [T_k,\infty)_{\mathbb{T}}$. Hence

$$\begin{aligned}\frac{x^{\Delta\sigma}(t)}{x^{\Delta}(t)} &\leqslant 1 + \mu(t)\frac{1}{\sqrt{k}\,t}\\ &= \frac{\sqrt{k}\,t + \sigma(t) - t}{\sqrt{k}\,t} \leqslant \frac{\sigma(t)}{\sqrt{k}\,t}.\end{aligned}$$

Hence, we have

$$\frac{x^{\Delta}(t)}{x^{\Delta\sigma}(t)} \geqslant \frac{\sqrt{k}\,t}{\sigma(t)},$$

and so we have

$$\frac{x(t)}{x^{\Delta\sigma}(t)} = \frac{x(t)}{x^{\Delta}(t)} \cdot \frac{x^{\Delta}(t)}{x^{\Delta\sigma}(t)} \geqslant k\Psi(t),$$

for $t \in [T_k,\infty)_{\mathbb{T}}$. Hence (2.6) holds. Also,

$$\frac{x^{\Delta\Delta}(t)}{x^{\Delta\sigma}(t)} = \frac{x^{\Delta\Delta}(t)}{x^{\Delta}(t)} \cdot \frac{x^{\Delta}(t)}{x^{\Delta\sigma}(t)} \geqslant \frac{\sqrt{k}\,t}{\sigma(t)}w(t) \geqslant \frac{kt}{\sigma(t)}w(t),$$

and so (2.4) follows from (2.7). Furthermore, since $x^{\Delta\Delta}(t)$ is decreasing,

$$\frac{x^{\Delta\Delta}(t)}{x^{\Delta\sigma}(t)} \geqslant \frac{x^{\Delta\Delta\sigma}(t)}{x^{\Delta\sigma}(t)} = w^{\sigma}(t),$$

and so (2.5) follows from (2.7). This completes the proof of Lemma 5. □

Lemma 6. *Let x be a solution of* (1.1) *satisfying part* (I) *of Lemma* 1 *and let* $w(t)=\frac{x^{\Delta\Delta}(t)}{x^{\Delta}(t)}$. *Then* $w(t)$ *satisfies* $(t-t_1)w(t)<1$ *for* $t\in[t_1,\infty)_{\mathbb{T}}$ *and* $\lim_{t\to\infty} w(t)=0$.

Proof. From (2.3), we see that

$$\begin{aligned}
w^{\Delta}(t) &\leqslant -\frac{w^2(t)}{1+\mu(t)w(t)}\\
&= -\frac{x^{\Delta\Delta}(t)}{x^{\Delta\sigma}(t)}w(t) \quad \bigl(\text{by (2.8)}\bigr)\\
&\leqslant -\frac{x^{\Delta\Delta\sigma}(t)}{x^{\Delta\sigma}(t)}w(t) \quad \bigl(\text{since } x^{\Delta\Delta}(t) \text{ is decreasing}\bigr)\\
&= -w(t)w^{\sigma}(t),
\end{aligned}$$

for $t\in[t_1,\infty)_{\mathbb{T}}$, and so

$$\left(-\frac{1}{w(t)}\right)^{\Delta}=\frac{w^{\Delta}(t)}{w(t)w^{\sigma}(t)}\leqslant -1$$

for $t\in[t_1,\infty)_{\mathbb{T}}$. Therefore

$$\int_{t_1}^{t}\frac{w^{\Delta}(s)}{w(s)w^{\sigma}(s)}\Delta s\leqslant -\int_{t_1}^{t}\Delta s,$$

and so

$$-\frac{1}{w(t)}+\frac{1}{w(t_1)}\leqslant -(t-t_1),$$

which implies $(t-t_1)w(t)<1$, since $w(t_1)>0$. □

Lemma 7. *Let $x(t)$, $w(t)$, and $\Psi(t)$ be as in Lemma* 5. *Define*

$$p_*:=\liminf_{t\to\infty} t\int_{t}^{\infty}\Psi(s)p(s)\Delta s \quad \text{and} \quad q_*:=\liminf_{t\to\infty}\frac{1}{t}\int_{t_1}^{t}\sigma^2(s)\Psi(s)p(s)\Delta s, \tag{2.9}$$

$$r:=\liminf_{t\to\infty} t w(t), \qquad R:=\limsup_{t\to\infty} t w(t), \tag{2.10}$$

and

$$l_*:=\liminf_{t\to\infty}\frac{\sigma(t)}{t}, \qquad l^*:=\limsup_{t\to\infty}\frac{\sigma(t)}{t}. \tag{2.11}$$

Then $0\leqslant r\leqslant R\leqslant 1$, $1\leqslant l_*\leqslant l^*\leqslant\infty$, *and*

$$p_*\leqslant r-r^2, \qquad q_*\leqslant \min\bigl\{1-R,\, Rl^*-r^2l_*\bigr\}. \tag{2.12}$$

Proof. Multiplying (2.6) by $(\sigma(s))^2$, and integrating for $t\geqslant t_1\geqslant T_k$ gives

$$\int_{t_1}^{t}\bigl(\sigma(s)\bigr)^2 w^{\Delta}(s)\Delta s+k\int_{t_1}^{t}\bigl(\sigma(s)\bigr)^2\Psi(s)p(s)\Delta s+\int_{t_1}^{t}\frac{(\sigma(s))^2w^2(s)}{1+\mu(s)w(s)}\leqslant 0. \tag{2.13}$$

An integration by parts in (2.13) yields

$$t^2 w(t) - t_1^2 w(t_1) - \int_{t_1}^{t} \left(s^2\right)^{\Delta_s} w(s)\Delta s$$
$$+ k\int_{t_1}^{t} \left(\sigma(s)\right)^2 \Psi(s) p(s)\Delta s + \int_{t_1}^{t} \frac{(\sigma(s))^2 w^2(s)}{1+\mu(s)w(s)}\Delta s \leqslant 0.$$

Since $(s^2)^{\Delta_s} = s + \sigma(s)$, we obtain after rearranging

$$t^2 w(t) \leqslant t_1^2 w(t_1) - k\int_{t_1}^{t} \left(\sigma(s)\right)^2 \Psi(s) p(s)\Delta s + \int_{t_1}^{t} H\big(s, w(s)\big)\Delta s, \tag{2.14}$$

where

$$H\big(s, w(s)\big) = \big(s + \sigma(s)\big)w(s) - \frac{(\sigma(s))^2 w^2(s)}{1+\mu(s)w(s)}.$$

We claim that $H(s, w(s)) \leqslant 1$, for $s \in [t_1, \infty)_{\mathbb{T}}$. To see this observe that if we let

$$g(s,u) := \big(s + \sigma(s)\big)u - \frac{(\sigma(s))^2 u^2}{1+\mu(s)u},$$

then we have (since $s + \sigma(s) = 2\sigma(s) - \mu(s)$), after some simplification

$$g(s,u) = \frac{(2\sigma(s) - \mu(s))u(1+\mu(s)u) - (\sigma(s))^2 u^2}{1+\mu(s)u}$$
$$= \frac{(2\sigma(s) - \mu(s))u - s^2 u^2}{1+\mu(s)u}.$$

We note that if $\mu(s) = 0$, then the maximum of $g(s,u)$ (with respect to u) occurs at $u_0 := \frac{1}{s}$. Moreover in the case $\mu(s) > 0$, after some calculations, one finds that for fixed $s > 0$, the maximum of $g(s,u)$ for $u \geqslant 0$ occurs at $u_0 = \frac{1}{s}$ also. Hence, we have

$$g(s,u) \leqslant g(s,u_0) = \big(s + \sigma(s)\big)u_0 - \frac{(\sigma(s))^2 u_0^2}{1+\mu(s)u_0}$$
$$= \frac{s+\sigma(s)}{s} - \frac{(\sigma(s))^2}{s(s+\mu(s))} = 1,$$

for $u \geqslant 0$. Hence we conclude that $H(s, w(s)) \leqslant 1$ and so

$$\int_{t_1}^{t} H\big(s, w(s)\big)\Delta s \leqslant t - t_1.$$

Substituting this in (2.14) and dividing by t we obtain

$$t w(t) \leqslant \frac{t_1^2 w(t_1)}{t} - \frac{k}{t}\int_{t_1}^{t} \left(\sigma(s)\right)^2 \Psi(s) p(s)\Delta s + \left(1 - \frac{t_1}{t}\right). \tag{2.15}$$

If we take the lim sup of both sides of (2.15) we get

$$R \leqslant 1 - kq_*.$$

Thus we have $R \leqslant 1 - kq_*$ for all $0 < k < 1$ and so we have

$$R \leqslant 1 - q_*.$$

Integrating (2.5) from t to ∞ we get

$$w(t) \geqslant k \int_t^\infty \Psi(s)p(s)\Delta s + \int_t^\infty w(s)w^\sigma(s)\Delta s \quad \text{for } t \in [T_k, \infty)_{\mathbb{T}}, \tag{2.16}$$

where $0 < k < 1$, is arbitrary. Hence, from (2.16) we have

$$\begin{aligned} tw(t) &\geqslant kt \int_t^\infty \Psi(s)p(s)\Delta s + t \int_t^\infty w(s)w^\sigma(s)\Delta s \\ &= kt \int_t^\infty \Psi(s)p(s)\Delta s + t \int_t^\infty \frac{(sw(s))(\sigma(s)w^\sigma(s))}{s\sigma(s)}\Delta s. \end{aligned} \tag{2.17}$$

Now for any $\epsilon > 0$ there exists $t_2 \geqslant t_1$ such that

$$r - \epsilon < tw(t),$$

for all $t \in [t_2, \infty)_{\mathbb{T}}$. Therefore, from (2.17) we get

$$\begin{aligned} tw(t) &\geqslant kt \int_t^\infty \Psi(s)p(s)\Delta s + t \int_t^\infty \frac{(r-\epsilon)^2}{s\sigma(s)}\Delta s \\ &\geqslant kt \int_t^\infty \Psi(s)p(s)\Delta s + (r-\epsilon)^2 t \int_t^\infty \frac{\Delta s}{s\sigma(s)} \\ &\geqslant kt \int_t^\infty \Psi(s)p(s)\Delta s + (r-\epsilon)^2 t \int_t^\infty \left(-\frac{1}{s}\right)^{\Delta_s} \Delta s \\ &\geqslant kt \int_t^\infty \Psi(s)p(s)\Delta s + (r-\epsilon)^2. \end{aligned} \tag{2.18}$$

Therefore, taking the lim inf of both sides of (2.18) gives

$$r \geqslant kp_* + (r-\epsilon)^2.$$

Since $\epsilon > 0$ is arbitrary and $0 < k < 1$ is arbitrary, we obtain

$$r \geqslant p_* + r^2.$$

Finally, we show that

$$q_* \leqslant Rl^* - r^2 l_*.$$

To see this let $\epsilon > 0$ be given, then there exists a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that

$$r - \epsilon < tw(t) < R + \epsilon, \quad l_* - \epsilon \leqslant \frac{\sigma(t)}{t} \leqslant l^* + \epsilon, \quad t \in [t_2, \infty)_{\mathbb{T}}.$$

Similar to how we used (2.5) to obtain (2.14) we can use (2.4) to obtain the inequality

$$\begin{aligned}
tw(t) &\leqslant \frac{t_2^2 w(t_2)}{t} - \frac{k}{t}\int_{t_2}^{t} \big(\sigma(s)\big)^2 \Psi(s) p(s)\,\Delta s \\
&\quad + \frac{1}{t}\int_{t_2}^{t} \big(s + \sigma(s)\big) w(s)\,\Delta s - \frac{k}{t}\int_{t_2}^{t} s\sigma(s) w^2(s)\,\Delta s \\
&= \frac{t_2^2 w(t_2)}{t} - \frac{k}{t}\int_{t_2}^{t} \big(\sigma(s)\big)^2 \Psi(s) p(s)\,\Delta s \\
&\quad + \frac{1}{t}\int_{t_2}^{t} \left(1 + \frac{\sigma(s)}{s}\right)\big(sw(s)\big)\,\Delta s - k\frac{1}{t}\int_{t_2}^{t} \frac{\sigma(s)}{s}\big(s^2 w^2(s)\big)\,\Delta s \\
&\leqslant \frac{t_2^2 w(t_2)}{t} - \frac{k}{t}\int_{t_2}^{t} \big(\sigma(s)\big)^2 \Psi(s) p(s)\,\Delta s \\
&\quad + (R + \epsilon)\frac{1}{t}\int_{t_2}^{t} \left(1 + \frac{\sigma(s)}{s}\right)\Delta s - k(r - \epsilon)^2 \frac{1}{t}\int_{t_2}^{t} \frac{\sigma(s)}{s}\,\Delta s \\
&\leqslant \frac{t_2^2 w(t_2)}{t} - \frac{k}{t}\int_{t_2}^{t} \big(\sigma(s)\big)^2 \Psi(s) p(s)\,\Delta s \\
&\quad + (R + \epsilon)(1 + l^* + \epsilon)\frac{(t - t_2)}{t} - k(r - \epsilon)^2 (l_* - \epsilon)\frac{(t - t_2)}{t}.
\end{aligned}$$

Taking the lim sup of both sides we get

$$R \leqslant -kq_* + (R + \epsilon)(1 + l^* + \epsilon) - k(r - \epsilon)^2 (l_* - \epsilon).$$

Since $\epsilon > 0$ is arbitrary we get

$$R \leqslant -kq_* + R(1 + l^*) - kr^2 l_*$$

and since $0 < k < 1$ is also arbitrary we have finally that

$$R \leqslant -q_* + R(1 + l^*) - r^2 l_*,$$

which yields the desired result. □

As a consequence of the previous lemmas, we may now establish some oscillation criteria.

Theorem 2. *Assume that* (2.1) *holds and let* $x(t)$ *be a solution of* (1.1). *If*

$$p_* = \liminf_{t\to\infty} t \int_t^\infty \Psi(s)p(s)\Delta s > \frac{1}{4}, \tag{2.19}$$

then $x(t)$ *is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of Eq. (1.1) with $x(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Then if part (I) of Lemma 1 holds, let $w(t)$ be as defined in Lemma 5. From Lemma 7 we obtain

$$p_* \leqslant r - r^2 \leqslant \frac{1}{4}$$

which contradicts (2.19). Now if part (II) of Lemma 1 holds, then by Lemma 3, $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Theorem 3. *Assume that* (2.1) *holds and let* $x(t)$ *be a solution of* (1.1). *If*

$$q_* = \liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^t \sigma(s)h_2(s,t_0)p(s)\Delta s > \frac{l^*}{1+l^*}, \tag{2.20}$$

then $x(t)$ *is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. From Lemma 7 we have that

$$q_* \leqslant \min\{1-R, Rl^* - r^2 l_*\} \leqslant \min\{1-R, Rl^*\}$$

which implies that $q_* \leqslant \frac{l^*}{1+l_*}$, which is a contradiction to (2.20). □

Theorem 4. *Assume* (2.1) *holds,* $0 \leqslant p_* \leqslant \frac{1}{4}$, *and*

$$q_* > \frac{l^* - (\frac{1}{2} - p_* - \frac{1}{2}\sqrt{1-4p_*})\, l_*}{1+l^*}. \tag{2.21}$$

Then every solution of (1.1) *is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. First we use the fact that $a := p_* \leqslant r - r^2$ to get that

$$r \geqslant r_0 := \frac{1}{2} - \frac{\sqrt{1-4a}}{2},$$

and so using (2.12),

$$\begin{aligned} q_* &\leqslant \min\{1-R, l^*R - r^2 l_*\} \\ &\leqslant \min\{1-R, l^*R - r_0^2 l_*\}, \end{aligned}$$

for $r_0 \leqslant R \leqslant 1$. Note that

$$1 - R = l^*R - r_0^2 l_*,$$

when

$$R = R_0 := \frac{1 + r_0^2 l_*}{1+l^*},$$

and so

$$\begin{aligned} q_* &\leqslant 1 - R_0 \\ &= 1 - \frac{1 + r_0^2 l_*}{1 + l^*} \\ &= \frac{l^* - (\frac{1}{2} - p_* - \frac{1}{2}\sqrt{1 - 4p_*})l_*}{1 + l^*}, \end{aligned}$$

after some easy calculations. This contradicts (2.21) and the proof is complete. □

Remark 2. A close look at the proof of Lemma 7 shows that the inequality

$$q_* \leqslant R l^* - r^2 l_*$$

holds, when we replace l^* and l_*, by

$$\lambda^* := \limsup_{t\to\infty} \frac{1}{t} \int_{t_1}^{t} \frac{\sigma(s)}{s} \Delta s \quad \text{and} \quad \lambda_* := \limsup_{t\to\infty} \frac{1}{t} \int_{t_1}^{t} \frac{\sigma(s)}{s} \Delta,$$

respectively. Then Theorems 3 and 4 hold with l^* and l_* replaced by λ^* and λ_*, respectively.

Remark 3. We note here that our methods of proof can be applied to the third-order linear equation

$$x^{\Delta\Delta\Delta} + p(t)x^\sigma = 0 \tag{2.22}$$

which also can be viewed as a generalization of the third-order differential equation

$$x''' + p(t)x = 0.$$

In particular, in Lemma 5 we can prove (2.3) with the coefficient of $p(t)$ replaced by $\frac{x^\sigma(t)}{x^{\Delta\sigma}(t)}$. Also we get (2.4)–(2.6), with $\Psi(t)$ replaced by $\frac{h_2(\sigma(t),t_0)}{\sigma(t)}$. Finally we get that Theorems 2–4 hold with p_* and q_* replaced by

$$\hat{p} = \liminf_{t\to\infty} t \int_{t_0}^{t} \frac{h_2(\sigma(s), t_0)}{\sigma(s)} p(s)\Delta s,$$

and

$$\hat{q} = \liminf_{t\to\infty} \frac{1}{t} \int_{t_1}^{t} \sigma(s) h_2\big(\sigma(s), t_0\big) p(s)\Delta s,$$

respectively.

3. Examples

Example 1. For examples where condition (2.19) in Theorem 2 is satisfied we get the following results. If $\mathbb{T} = [0, \infty)$, then $h_2(t, 0) = \frac{t^2}{2}$ and $\Psi(t) = \frac{h_2(t,t_0)}{t} = \frac{t}{2}$ so (2.19) holds if

$$\liminf_{t\to\infty} t \int_{t}^{\infty} s p(s)\, ds > \frac{1}{2}.$$

If $\mathbb{T} = \mathbb{N}_0$, then $h_2(t,0) = \frac{1}{2}t^{\underline{2}}$, so (2.19) holds if

$$\liminf_{n\to\infty} n \sum_{k=n}^{\infty} kp(k) > \frac{1}{2}.$$

If $\mathbb{T} = q^{\mathbb{N}_0}$, then $h_2(t,1) = \frac{(t-1)(t-q)}{1+q}$ and $\sigma(t) = qt$ so (2.19) holds if

$$\liminf_{t\to\infty} t \int_t^{\infty} sp(s)\Delta s > \frac{q(1+q)}{4}.$$

Example 2. For examples where condition (2.20) in Theorem 3 is satisfied we get the following results. If $\mathbb{T} = [0,\infty)$, then (2.20) holds if

$$\liminf_{t\to\infty} \frac{1}{t} \int_{t_1}^{t} s^3 p(s)\,ds > 1.$$

If $\mathbb{T} = \mathbb{N}_0$, then (2.20) holds if

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=n_1}^{n-1} k^3 p(k) > 1.$$

If $\mathbb{T} = q^{\mathbb{N}_0}$, then (2.20) holds if

$$\liminf_{t\to\infty} \frac{1}{t} \int_{t_1}^{t} s^3 p(s)\Delta s > 1.$$

Example 3. Consider the third-order dynamic equation

$$x^{\Delta\Delta\Delta}(t) + \frac{\beta}{t^3}x(t) = 0, \tag{3.1}$$

for $t \in \mathbb{T} := [1,\infty)$. Here $p(t) = \frac{\beta}{t^3}$. To apply Theorem 2 it is easy to show that (2.1) holds and $p_* = \frac{\beta}{2}$. Hence, by Theorem 2, if $\beta > \frac{1}{2}$, then every solution of (3.1) is oscillatory or converges to zero. As a specific example note that if $\beta = 6$, then a basis of the solution space of (3.1) is given by

$$\{t^{-1},\ t^2\cos(\sqrt{2}\log t),\ t^2\sin(\sqrt{2}\log t)\},$$

which contains oscillatory solutions and satisfies the property that every nonoscillatory solution converges to zero.

We wish to consider next two examples illustrating condition (2.21).

Example 4. Let $\mathbb{T} = q^{\mathbb{N}_0}$ and let

$$p(t) := \frac{\alpha}{th_2(t,1)}, \quad 0 < \alpha \leqslant \frac{1}{4}.$$

Then we have $\Psi(t)p(t) = \frac{\alpha}{t\sigma(t)}$, so

$$p_* = \liminf_{t\to\infty} t \int_t^\infty \Psi(s)p(s)\Delta s$$
$$= \alpha \liminf_{t\to\infty} t \int_t^\infty \frac{\Delta s}{s\sigma(s)} = \alpha,$$

and since $(\sigma(t))^2\Psi(t)p(t) = \alpha q$, we have

$$q_* = \liminf_{t\to\infty} \frac{1}{t} \int_{t_1}^t \big(\sigma(s)\big)^2 \Psi(s)p(s)\Delta s$$
$$= \liminf_{t\to\infty} \frac{1}{t} \int_{t_1}^t \alpha q \Delta s$$
$$= \alpha q > q = p_*.$$

Since $l^* = l_* = q$, we see that if $q_* > \frac{q}{1+q}$, then Theorem 3 applies. That is, if $\alpha > \frac{1}{1+q}$, all solutions are oscillatory or converge to zero. If $0 < \alpha \leqslant \frac{1}{1+q}$, then condition (2.21) of Theorem 4 is equivalent to

$$q_* = \alpha q > \frac{q - (\frac{1}{2} - \alpha - \frac{1}{2}\sqrt{1-4\alpha})q}{1+q},$$

which in turn is equivalent to

$$\alpha > \frac{(\frac{1}{2} + \alpha + \frac{1}{2}\sqrt{1-4\alpha})}{1+q}.$$

Solving this inequality gives

$$q > \frac{1+\sqrt{1-4\alpha}}{2\alpha}. \tag{3.2}$$

Therefore, for any $0 < \alpha \leqslant \frac{1}{1+q}$, Theorem 4 implies that all solutions are oscillatory or converge to zero if (3.2) holds. For example, if $\alpha = \frac{1}{8}$ and $q > 4 + \frac{4}{\sqrt{2}} \approx 6.82$, then Theorem 4 applies and Theorem 3 does not apply if $4 + \frac{4}{\sqrt{2}} < q < 7$.

Example 5. We let $\mathbb{T} = q^{\mathbb{N}_0} \cup aq^{\mathbb{N}_0}$ where $1 < a < q < a^2$. Then

$$\mathbb{T} = \{1, a, q, aq, q^2, aq^2, \ldots\}.$$

Thus, $t_{2n} = q^n$, and $t_{2n+1} = aq^n$, for $n = 0, 1, 2, \ldots$, so

$$\frac{\sigma(t)}{t} = \begin{cases} \frac{q}{a}, & t = t_{2n+1}, \\ a, & t = t_{2n}, \end{cases}$$

and so $l^* = a$ and $l_* = \frac{q}{a}$. We have

$$\frac{l^*}{1+l^*} = \frac{a}{1+a},$$

and so if $q_* < \frac{a}{1+a}$, we cannot apply Theorem 3. Likewise, if $p_* < \frac{1}{4}$, Theorem 2 does not apply. Therefore, if $p_* = \frac{1}{8}$ and with $l^* = \frac{3}{2}$, $l_* = \frac{4}{3}$, and

$$\frac{l^* - (\frac{1}{2} - p_* - \frac{1}{2}\sqrt{1-4p_*})l_*}{1+l^*} \approx 0.588 < 0.6 = \frac{a}{1+a}.$$

So if $0.588 < q_* < 0.6$ and $p_* = \frac{1}{8}$, then Theorem 4 applies but Theorems 2 and 3 do not.

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